Prof. I.K. Rana
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Lecture no 16

Measure Theory

IK Rana

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$(X, \mathcal{S})$ measurable space

$f : X \longrightarrow \mathbb{R}^*$

f mble $\iff$ $f^{-1}(I) \in \mathcal{S}$ $\forall$ interval $I \subseteq \mathbb{R}$.

$\iff$ $f^{-1}([c, +\infty)) \in \mathcal{S}$ $\forall$ $c \in \mathbb{R}$

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f_1, f_2 mble $\Rightarrow$ $f_1 + f_2$

$\Rightarrow$ $f_1 f_2$ is mble

$$f_n : X \longrightarrow \mathbb{R}^*$$

f_n measurable $\forall\, n \geq 1$.

$$\left(\bigvee_{n=1}^{\infty} f_n\right)(x) := \max\{f_n(x) \mid n \geq 1\}$$

To show $\left(\bigvee_{n=1}^{\infty} f_n\right)$ is measurable.

$$\left(\bigvee_{n=1}^{\infty} f_n\right)^{-1}\left([c, +\infty)\right) = \left\{x \in X \,\middle|\, \left(\bigvee_{n=1}^{\infty} f_n\right)(x) \geq c\right\}$$

$$\underbrace{\left(\bigvee_{n=1}^{\infty} f_n\right)^{-1}\left[(-\infty, c]\right]}_{\in \mathcal{S}} = \left\{x \in X \,\middle|\, \left(\bigvee_{n=1}^{\infty} f_n\right)(x) \leq c\right\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in X \mid f_n(x) \leq c\}}_{\in \mathcal{S}}$$

$$\left(\bigwedge_{n=1}^{\infty} f_n\right)(x) = \min\{f_n(x) \mid \cancel{x}\; n \geqslant 1\}$$

$$c \in \mathbb{R}$$

$$\left(\bigwedge_{n=1}^{\infty} f_n\right)^{-1}\left([c,+\infty)\right) = \left\{x \in X \;\middle|\; \min_{n \geqslant 1} f_n(x) \geqslant c\right\}$$

$$= \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geqslant c\}$$

$$= \bigcap_{n=1}^{\infty} f_n^{-1}\left([c,+\infty)\right)$$

$$\in \mathcal{S}.$$

$\{f_n\}_{n\geq 1} \quad f_n : X \longrightarrow \mathbb{R}^*$

$\forall \; x \in X$

$$(\limsup f_n)(x) = \inf_m \{ \sup \{ f_n(x) \mid n \geq m \} \}$$

$$(\liminf f_n)(x) := \sup_{m \geq 1} \{ \inf \{ f_n(x) \mid n \geq m \} \}$$

$$(\limsup f_n)(x) \geq (\liminf f_n)(x)$$

$f_n(x) \longrightarrow \underline{f(x)}$ iff $(\limsup f_n)(x)$

$= f(x) = (\liminf f_n)(x)$

$$\left(\limsup f_n\right)(x) = \underline{\inf_{m \geq 1}} \left\{ \underline{\sup_{n \geq m} \left(f_n(x) \right)} \right\}$$

$\limsup f_n$ is measurable

$\liminf f_n$ is measurable

$$f_n \longrightarrow f = \begin{cases} \limsup f_n \\ \liminf f_n \end{cases}$$

f, g are extended Real Valued.

$$(f+g)(x) = \underline{f(x) + g(x)}$$

Problem $\left.\begin{cases} f(x) = +\infty \\ g(x) = -\infty \end{cases} \text{ or } \begin{cases} f(x) = -\infty \\ g(x) = +\infty \end{cases}\right\}$

$$A = \left\{ x \in X \;\middle|\; \begin{array}{l} f(x) = +\infty,\ g(x) = -\infty \\ \text{or } f(x) = -\infty,\ g(x) = +\infty \end{array} \right\}$$

$$A \in \mathcal{S}$$

$$(f+g)(x) = \begin{cases} f(x)+g(x) & \text{if } x \notin A \\ \alpha & \text{if } x \in A \end{cases}$$

Ex $f+g$ is Σ- measurable.

$f, g : X \longrightarrow \mathbb{R}^*$, mble

Tosh me

$\{x \in X \mid f(x) < g(x)\} \in \mathcal{S}$?

$\forall x \in X$, $\exists$ rational r such that $f(x) < r < g(x)$

$$\{x \in X \mid f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x \in X \mid f(x) < r < g(x)\}$$

$$= \bigcup_{r \in \mathbb{Q}} \left(\{x \in X \mid f(x) < r\} \cap \{x \in X \mid g(x) > r\} \right)$$

$$\{x \in X \mid f(x) < g(x)\}$$

$$= \underbrace{\bigcup_{r \in \mathbb{Q}} \Big(\underbrace{f^{-1}([-\infty, r))}_{\in \mathcal{S}} \cap \underbrace{g^{-1}\{(r, +\infty]\}}_{\in \mathcal{S}} \Big)}_{\in \mathcal{S}}$$

$$\Rightarrow \{x \in X \mid f(x) < g(x)\}^c$$

$$= \{x \in X \mid f(x) \geq g(x)\}$$

$$\{x \in X \mid f(x) > g(x)\}$$

$$= \bigcup_{r \in \mathbb{Q}} \left(f^{-1}(r, +\infty] \cap \left(g^{-1}[-\infty, r) \right) \right)$$

$$\in S$$

$$\{x \in X \mid f(x) \leq g(x)\} \in S$$

$$f : X \longrightarrow \mathbb{R}^*$$

$$g : X \longrightarrow \mathbb{R}^*$$

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

Suppose f is Σ- measurable.

To show g is Σ- measurable?

$$g^{-1}(I) \in \Sigma \quad \forall \;, I \subseteq \mathbb{R}^*$$

$$g^{-1}(I) = \{x \in X \mid g(x) \in I\}$$

$$g^{-1}(I) = \Big(\{x \in X \mid g(x) \in I\} \cap A\Big)$$

$$\cup \{x \in X \mid g(x) \in I\} \cap A^c\}$$

where $A = \{x \in X \mid f(x) \neq g(x)\}$

$\underline{\mu(A) = 0} \Rightarrow A \in \mathcal{S}$

$$\Rightarrow \underline{\{x \in X \mid g(x) \in I\} \cap A} \in \mathcal{S}$$

$(\because (X, \mathcal{S}, \mu)$ is complete$)$

$$\{x \in X \mid g(x) \in I\} \cap A^c = \{x \in X \mid f(x) \in I\} \cap A^c$$

$$\overset{-1}{g}(I) = (\overset{-1}{g}(I) \cap A) \cup (\overset{-1}{g}(I) \cap A^c)$$

$$= \overset{-1}{g}(I \cap A) \cup (\overset{-1}{f}(I) \cap A^c)$$

$$\mu(\overset{-1}{g}(I \cap A)) = 0 \qquad \overset{-1}{f}(I) \cap A^c \in \mathcal{S}$$

$$\Downarrow$$

$$\overset{-1}{g}(I \cap A) \in \mathcal{S}$$

$$\Longrightarrow \quad \overset{-1}{g}(I) \in \mathcal{S}.$$

f mble, $f = g$ a.e. (μ) $\Rightarrow$ g mble

$(X, \mathcal{S}, \mu)$ — complete measure space

$f_n : X \longrightarrow \mathbb{R}^*$, mble

$f_n \longrightarrow f$ a.e. (μ)

$A = \{x \in X \mid f_n(x) \not\to f(x)\}$

Given $A \in \mathcal{S}$ and $\mu(A) = 0$.

To show f is mble.

$$f^{-1}(I) = \left(\underline{f^{-1}(I) \cap A}\right) \cup \left(\underline{f^{-1}(I) \cap A^c}\right)$$

$\underline{f^{-1}(I) \cap A}$ is a subset of A, so $f^{-1}(I) \cap A \in \mathcal{S}$.

$f^{-1}(I) \cap A^c = \left(\lim_{n \to \infty} f_n\right)^{-1}(I) \cap A^c \in \mathcal{S}$

$\cancel{f \Rightarrow}$ ———

$f : \mathbb{R} \longrightarrow \mathbb{R}$, f continuous.

<u>Claim</u> f is Borel measurable.

i.e. $f^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$ $\forall$ $E \in \mathcal{B}_{\mathbb{R}}$

$\mathcal{A} = \{ \underline{E \in \mathcal{B}_{\mathbb{R}}} \mid f^{-1}(E) \in \mathcal{B}_{\mathbb{R}} \}$

<u>To show</u> $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$.

<u>Note</u> $\mathcal{A} \subseteq \mathcal{B}_{\mathbb{R}}$

(i) Open sets $\subseteq \mathcal{A}$

(ii) $\mathcal{A}$ is a σ-algebra

(i) let $\underline{U \subseteq \mathbb{R} \text{ be open}}$

f continuous $\Rightarrow f^{-1}(U)$ is open

$\Rightarrow \underline{f^{-1}(U) \in \mathcal{B}_{\mathbb{R}}}$

$\Rightarrow$ Open sets $\subseteq \underline{\mathcal{A}}$.

(ii) $\phi = f^{-1}(\phi)$, $\mathbb{R} = f^{-1}(\mathbb{R}) \in \mathcal{A}$

$E \in \mathcal{A} \Rightarrow f^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$

$\Rightarrow (f^{-1}(E))^c \in \mathcal{B}_{\mathbb{R}}$

$\Rightarrow f^{-1}(E^c) \in \mathcal{B}_{\mathbb{R}}$

$\Rightarrow E^c \in \mathcal{A}$

$E_n \in \mathcal{A}, \quad n \geq 1$

$$\Rightarrow \quad f^{-1}(E_n) \in \mathcal{B}_{\mathbb{R}}$$

$$\Rightarrow \quad \bigcup_{n=1}^{\infty} f^{-1}(E_n) \in \mathcal{B}_{\mathbb{R}}$$

$$\qquad \qquad \quad \|$$

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) \in \mathcal{B}_{\mathbb{R}}$$

$$\Rightarrow \quad \bigcup_{n=1}^{\infty} E_n \in \cancel{\mathcal{B}_{\mathbb{R}}}\ \mathcal{A}.$$

Hence $\mathcal{A}$ is a σ-algebra

$$f : \mathbb{R} \longrightarrow \mathbb{R}^*$$

$$f = \chi_A \quad , \quad A \subseteq X$$

χ_A is Lebesgue measurable

iff $A \in \mathcal{L}_{\mathbb{R}}$